HOSSAM GHANEM

(5) 3.5 *Techniques for finding limits of Trigonometric function (B)*

Example 1 33 October 25,2001 A	Find the limit , if it exists $\lim_{x\to 0}\dfrac{1-\cos(3x)}{\sin^2(x)}$

Solution

$$L = \lim_{x\to 0}\frac{1-\cos(3x)}{\sin^2(x)} = \frac{1-1}{0} = \frac{0}{0}$$

$$L = \lim_{x\to 0}\frac{1-\cos(3x)}{\sin^2(x)} = \lim_{x\to 0}\frac{2\sin^2 6x}{\sin^2(x)} = \lim_{x\to 0}\frac{2\dfrac{\sin^2 6x}{x^2}}{\dfrac{\sin^2(x)}{x^2}} = \lim_{x\to 0}\frac{2\cdot 36\dfrac{\sin^2 6x}{(6x)^2}}{\dfrac{\sin^2(x)}{x^2}} = \frac{2\cdot 36\cdot 1}{1} = 72$$

Example 2 16 June 6 .1996	Find the limit , if it exists $\lim_{x\to 0}\dfrac{\cos x-1}{x\sin x}$

Solution

$$L = \lim_{x\to 0}\frac{\cos x-1}{x\sin x} = \frac{1-1}{0} = \frac{0}{0}$$

$$L = \lim_{x\to 0}\frac{\cos x-1}{x\sin x} = \lim_{x\to 0}\frac{2\sin^2(x/2)}{x\sin x} = \lim_{x\to 0}\frac{2\cdot\dfrac{x^2}{4}\dfrac{\sin^2(x/2)}{(x/2)^2}}{x\cdot x\dfrac{\sin x}{x}}$$

$$= \lim_{x\to 0}\frac{2\cdot x^2}{4x^2}\cdot\frac{\left(\dfrac{\sin(x/2)}{(x/2)}\right)^2}{\left(\dfrac{\sin x}{x}\right)} = \frac{2}{4}\cdot\frac{(1)^2}{1} = \frac{1}{2}$$

Example 3 3 December 30, 1991	Find the following limit , if it exists $\lim_{x\to\infty} x\sin\left(\dfrac{1}{x}\right)$

Solution

$$L = \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \infty\cdot 0$$

let $t = \dfrac{1}{x}$ $\quad\therefore$ as $x\to\infty$, $t\to 0$

$$L = \lim_{t\to 0}\frac{\sin t}{t} = 1$$

Example 4

Evaluate the following limit $\lim_{x\to\frac{\pi}{2}} \dfrac{\cos x}{\pi - 2x}$

Solution

$$L = \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\pi - 2x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{2\left(\frac{\pi}{2} - x\right)} = \frac{1}{2} \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x} = \frac{0}{0}$$

$$L = \frac{1}{2} \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x}$$

let $t = \frac{\pi}{2} - x$ $\quad \therefore x = \frac{\pi}{2} - t$ $\quad \therefore$ as $x \to \frac{\pi}{2}$, $t \to 0$

$$L = \frac{1}{2} \lim_{t\to 0} \frac{\cos\left(\frac{\pi}{2} - t\right)}{t} = \frac{1}{2} \lim_{t\to 0} \frac{\sin t}{t} = \frac{1}{2}(1) = \frac{1}{2}$$

Example 5

54 November 16, 2009 A

Find the limit , if it exists $\lim_{x\to -2} \dfrac{\sin(x+2)}{x^3+8}$

Solution

$$L = \lim_{x\to -2} \frac{\sin(x+2)}{x^3+8} = \frac{\sin 0}{-8+8} = \frac{0}{0}$$

$$L = \lim_{x\to -2} \frac{\sin(x+2)}{x^3+8} = \lim_{x\to -2} \frac{\sin(x+2)}{(x+2)} \cdot \frac{1}{(x^2-2x+4)}$$

$$L_1 = \lim_{x\to -2} \frac{1}{(x^2-2x+4)} = \frac{1}{4+4+4} = \frac{1}{12}$$

$$L_2 = \lim_{x\to -2} \frac{\sin(x+2)}{(x+2)}$$

let $t = x + 2$ $\quad$ as $\quad x \to -2$, $\quad t \to 0$

$$L_2 = \lim_{t\to 0} \frac{\sin t}{t} = 1$$

$$L = L_1 \cdot L_2 = 1 \cdot \frac{1}{12} = \frac{1}{12}$$

Example 6

6 April 8, 1993

Find the limit , if it exists $\lim_{x\to -1} \dfrac{x^4 - x^3 + x - 1}{\sin(x+1)}$

Solution

$$L = \lim_{x\to -1} \frac{x^4 - x^3 + x - 1}{\sin(x+1)} = \frac{1+1-1-1}{\sin 0} = \frac{0}{0}$$

$$x^4 - x^3 + x - 1 = (x^4 - x^3) + (x - 1) = x^3(x-1) + (x-1) = (x-1)(x^3+1) = (x-1)(x+1)(x^2-x+1)$$

$$L = \lim_{x\to -1} \frac{x^4 - x^3 + x - 1}{\sin(x+1)} = \lim_{x\to -1} \frac{(x-1)(x+1)(x^2-x+1)}{\sin(x+1)} = \lim_{x\to -1} \frac{(x+1)}{\sin(x+1)} \cdot (x-1)(x^2-x+1)$$

$$L_1 = \lim_{x\to -1} (x-1)(x^2-x+1) = (-1-1)(1+1+1) = -6$$

$$L_2 = \lim_{x\to -1} \frac{(x+1)}{\sin(x+1)}$$

let $t = x + 1$ $\quad \therefore$ as $\quad x \to -1$, $\quad t \to 0$

$$L_2 = \lim_{t\to 0} \frac{t}{\sin t} = 1 \qquad \rightarrow \qquad L = L_1 \cdot L_2 = -6\,(1) = -6$$

Example 7 15 July 15, 1996	Find the following limit , if it exists $\lim_{x\to 1}\dfrac{1-\cos(x-1)}{(x-1)^2}$

Solution

$$L=\lim_{x\to 1}\frac{1-\cos(x-1)}{(x-1)^2}=\frac{1-1}{0}=\frac{0}{0}$$

$$L=\lim_{x\to 1}\frac{1-\cos(x-1)}{(x-1)^2}=\lim_{x\to 1}\frac{2\sin^2\left(\frac{x-1}{2}\right)}{4\left(\frac{x-1}{2}\right)^2}$$

let $t=\dfrac{x-1}{2}$ $\quad\therefore$ as $x\to 1$, $t\to 0$

$$L=\lim_{t\to 0}\frac{1}{2}\cdot\frac{\sin^2 t}{t^2}=\frac{1}{2}\lim_{t\to 0}\left(\frac{\sin t}{t}\right)^2=\frac{1}{2}\ (1)^2=\frac{1}{2}$$

Example 8 41 March 30, 2005	Evaluate the following limit $\lim_{x\to 2}(x-2)^2\cot^2(x-2).$

Solution

$$L=\lim_{x\to 2}(x-2)^2\cot^2(x-2)=0\cdot\infty$$

$$L=\lim_{x\to 2}\frac{(x-2)^2}{\tan^2(x-2)}$$

let $t=x-2$ $\quad\therefore$ as $x\to 2$, $t\to 0$

$$L=\lim_{t\to 0}\frac{t^2}{\tan^2 t}=\lim_{t\to 0}\left(\frac{t}{\tan t}\right)^2=(1)^2=1$$

Example 9 11 August 11, 1994 A	Find the limit , if it exists $\lim_{x\to 0}\dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}$

Solution

$$L=\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}=\frac{1-1}{0}=\frac{0}{0}$$

$$L=\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}\cdot\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$=\lim_{x\to 0}\frac{(1+\sin x)-(1-\sin x)}{\tan x}\cdot\frac{1}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}=\lim_{x\to 0}\frac{2\sin x}{\frac{\sin x}{\cos x}}\cdot\frac{1}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$=\lim_{x\to 0}\frac{2\sin x\cos x}{\sin x}\cdot\frac{1}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}=\lim_{x\to 0}\frac{2\cos x}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}=\frac{2}{1+1}=1$$

Example 10

30 October 19, 2000 A

Find the limit , if it exists $\lim_{x\to 0}\dfrac{\sqrt{1+\sin x}-1}{\tan x}$

Solution

$$L=\lim_{x\to 0}\frac{\sqrt{1+\sin x}-1}{\tan x}=\frac{1-1}{0}=\frac{0}{0}$$

$$L=\lim_{x\to 0}\frac{\sqrt{1+\sin x}-1}{\tan x}\cdot\frac{\sqrt{1+\sin x}+1}{\sqrt{1+\sin x}+1}=\lim_{x\to 0}\frac{(1+\sin x)-1}{\frac{\sin x}{\cos x}}\cdot\frac{1}{\sqrt{1+\sin x}+1}$$

$$=\lim_{x\to 0}\frac{\sin x\cos x}{\sin x}\cdot\frac{1}{\sqrt{1+\sin x}+1}=\lim_{x\to 0}\frac{\cos x}{\sqrt{1+\sin x}+1}=\frac{1}{1+1}=\frac{1}{2}$$

Example 11

10 June 6, 1994

Find the following limit , if it exists $\lim_{x\to 0}\left(\dfrac{1-\cos 2x}{x}+\dfrac{x}{\sin 3x}\right)$

Solution

$$L=\lim_{x\to 0}\left(\frac{1-\cos 2x}{x}+\frac{x}{\sin 3x}\right)=\frac{1-1}{0}+\frac{0}{0}$$

$$L=\lim_{x\to 0}\left[\frac{2(1-\cos 2x)}{2x}+\frac{\frac{1}{3}\cdot 3x}{\sin 3x}\right]=2\lim_{x\to 0}\frac{(1-\cos 2x)}{2x}+\frac{1}{3}\lim_{x\to 0}\frac{3x}{\sin 3x}=2\,(0)+\frac{1}{3}(1)=\frac{1}{3}$$

Example 12

Evaluate the following limit $\lim_{x\to 0}\dfrac{2\cos x+3x-2}{5x}$

Solution

$$L=\lim_{x\to 0}\frac{2\cos x+3x-2}{5x}=\frac{2+0-2}{0}=\frac{0}{0}$$

$$L=\lim_{x\to 0}\frac{2\cos x-2}{5x}+\frac{3x}{5x}=\lim_{x\to 0}\frac{2(\cos x-1)}{5x}+\lim_{x\to 0}\frac{3x}{5x}=\frac{2}{5}\lim_{x\to 0}\frac{\cos x-1}{x}+\lim_{x\to 0}\frac{3}{5}=\frac{2}{5}\,(0)+\frac{3}{5}=\frac{3}{5}$$

Example 13

13 November 13, 1995

33 January 20, 2009

Evaluate the following limit $\lim_{x\to 0}\dfrac{\sin 2x-2\sin x}{x^2}$

Solution

$$L=\lim_{x\to 0}\frac{\sin 2x-2\sin x}{x^2}=\frac{0}{0}$$

$$L=\lim_{x\to 0}\frac{2\sin x\cos x-2\sin x}{x^2}=\lim_{x\to 0}\frac{2\sin x(\cos x-1)}{x^2}=2\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\cos x-1}{x}=2(1)(0)=0$$

Example 14 36 January 17, 2010	Evaluate the following limit $\lim_{x\to 0}\frac{(3x-\tan x)^2}{x^2}$

Solution

$$L=\lim_{x\to 0}\frac{(3x-\tan x)^2}{x^2}=\frac{0}{0}$$

$$L=\lim_{x\to 0}\frac{9x^2-6x\tan x+\tan^2 x}{x^2}=\lim_{x\to 0}\frac{9x^2}{x^2}-\frac{6x\tan x}{x^2}+\frac{\tan^2 x}{x^2}$$

$$=\lim_{x\to 0}9-6\lim_{x\to 0}\frac{\tan x}{x}+\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2=9-6+1=4$$

Example 15 40 August 7, 2011	Evaluate the following limit $\lim_{x\to 0}\frac{\sin(x+1)}{x+1}$

Solution

$$L=\lim_{x\to 0}\frac{\sin(x+1)}{x+1}=\frac{\sin 1}{1}=\sin 1$$

Homework

1	Find the limit , if it exists	$\lim_{x\to 0}\frac{x^2}{1-\cos x}$	23 May 26. 2002
2	Evaluate the following limit (if it exists)	$\lim_{x\to 0}\frac{\sin 2x-2\sin x}{x\sin x}$	21 May 27. 2001
3	Find	$\lim_{x\to 0}\frac{x(\sin x)}{1-\cos 2x}$	
4	Find the limit , if it exists	$\lim_{x\to \pi}\frac{x-\pi}{\sin x}$	
5	Find the limit , if it exists	$\lim_{x\to 1}\frac{\sin(x-1)}{x^2-1}$	12 November 2, 1995
6	Evaluate the following limit(if it exists)	$\lim_{x\to 0}\frac{1-\cos(3x)}{x^2}$	19 November 1, 1997
7	Evaluate the following limit (if it exists)	$\lim_{x\to 1}\frac{[\tan 2(x-1)]^2}{x^2-2x+1}$	44 November 9, 2006
8	Find	$\lim_{x\to 0}\frac{\sqrt{1+\sin x}-1}{x}$	5 July 13, 1992
9	Find the limit , if it exists	$\lim_{x\to \infty} x\left(1-\cos\frac{1}{x}\right)$	
10	Find the limit , if it exists)	$\lim_{x\to 0}(\csc x-\cot x)$	29 June 4. 2007
11	Find the limit , if it exists	$\lim_{x\to 0}\frac{1+2\sin x-\cos x}{x}$	23 October 24, 1998

Homework

12	Find the limit , if it exists	$\lim_{x\to 0}\frac{1-\cos x}{x^2}$	
13	Evaluate the following limit (if it exists)	$\lim_{x\to 0}\frac{1-\cos x}{\tan x}$	
14	Find	$\lim_{x\to 0}\frac{\tan^3 2x}{x^4+2x^3}$	1 December 3, 1992
15	Evaluate the following limit (if it exists)	$\lim_{x\to 2}\frac{\sin(x-2)}{x^2-4}$	19 July 29, 2000
16	Find	$\lim_{x\to -3}\frac{1-\cos(x+3)}{x^2+x-6}$	20 January 3 ,2001
17	Evaluate the following limit (if it exists)	$\lim_{x\to 0}\frac{\sin 7x}{3x^4-5x}$	27 May 30. 2006
18	Find	$\lim_{x\to 2}\left(\frac{\sin(x-2)}{x^2+x-6}\right)$	37 June 6, 2010